

Paramagnetic alignment of thermally rotating dust

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ABSTRACT

Paramagnetic alignment of thermally rotating oblate dust grains is studied analytically for finite ratios of grain to gas temperatures. For such ratios, the alignment of angular momentum \mathbf{J} in respect to the grain axis of maximal inertia is only partial. We treat the alignment of \mathbf{J} using perturbative methods and disentangle the problem of \mathbf{J} alignment in grain body axes from that of \mathbf{J} alignment in respect to magnetic field. This enables us to find the alignment of grain axes to magnetic field and thus relate our theory to polarimetric observations. Our present results are applicable to the alignment of both paramagnetic and superparamagnetic grains.

Subject headings: dust, extinction — ISM, clouds — ISM, polarization

1. Introduction

Grain alignment in molecular clouds remains a puzzle in spite of intensive experimental research in the area (see Whittet 1992, Goodman et al. 1995, Dotson 1996, Hildebrand & Dragovan 1995). We believe, that one reason for this is that the theory of the alignment is still not adequate (see a recent review by Roberge 1996).

Paramagnetic alignment of thermally rotating grains² discovered by Davis-Greenstein in 1951 is often named a candidate for explaining the alignment in molecular clouds. Since then numerous studies of the alignment were performed (e.g. Jones & Spitzer 1967, Purcell 1969, Purcell & Spitzer 1971). However the important effect of internal relaxation was described by Purcell only in 1979. He found that the angular momentum \mathbf{J} is aligned with the grain axis of maximal moment of inertia, \mathbf{z}^b , (henceforth the axis of major inertia) on a time scale much shorter than the gaseous damping time. Both the recent numerical (Roberge, DeGraff & Flaherty 1993) and analytical (Lazarian 1995a, further on Paper I) studies of the alignment of thermally rotating oblate grains³ accounted for thermal relaxation by assuming \mathbf{J} and \mathbf{z}^b to be parallel. This assumption, however, is valid only when grain rotational temperature is much greater than its material temperature.

The fact that for finite ratios of the said temperatures the internal alignment is not perfect was pointed out in Lazarian (1994), while the quantitative study of this effect was given in Lazarian & Roberge (1997) (henceforth LR). The purpose of the present paper is to incorporate the effect of incomplete internal relaxation into the theory of paramagnetic alignment of oblate grains by using perturbative approach.

In Section 2, we formulate the problem. In Sections 3 and 4 we determine the Fokker-Planck coefficients arising from grain-gas interactions, while magnetic coefficients and the Fokker-Planck equation for the Barnett relaxation are reproduced in Section 5. The iterations to the measure of grain axis alignment are obtained in Section 6, and the summary of main results is presented in Section 7.

²A term *thermally rotating* means that a grain is rotating at Brownian velocities. This is possible whenever grains are not subjected to uncompensated torques (Purcell 1979, Draine & Weingartner 1996a,b).

³The correspondence between the analytical and numerical results was established in DeGraff, Roberge & Flaherty (1997).

2. Formulation of the problem

Starlight polarization is caused by the alignment of grain axes, while dynamical evolution is defined in terms of angular momentum. The importance of relating these two different quantities was realized by researchers very early (see Davis & Greenstein 1951, Davis 1955, Jones & Spitzer 1967, Purcell & Spitzer 1971). In the present paper we relate these quantities when \mathbf{J} is partially aligned with the grain axis of major inertia.

For an ensemble of spheroidal grains the measure of axis alignment in respect to magnetic field can be described by the Rayleigh reduction factor (Greenberg 1968)

$$R = \frac{3}{2} \langle \cos^2 \phi - \frac{1}{3} \rangle \quad , \quad (1)$$

where ϕ is the angle between the axis of an oblate grain and magnetic field \mathbf{H} and, here and further on, angular brackets $\langle \rangle$ denote the ensemble averaging. Similarly, the alignment of angular momentum in grain body axes is given by

$$Q_X = \frac{3}{2} \langle \cos^2 \theta - \frac{1}{3} \rangle \quad , \quad (2)$$

where θ is the angle between the axis of major inertia and \mathbf{J} . The alignment of \mathbf{J} in respect to magnetic field is characterized by

$$Q_J = \frac{3}{2} \langle \cos^2 \beta - \frac{1}{3} \rangle \quad , \quad (3)$$

where β is the angle between \mathbf{J} and \mathbf{H} . These three measures are not independent. Indeed, it is obvious from spherical trigonometry (see eq. 108, Davis & Greenstein 1951), that

$$\langle \cos^2 \phi \rangle = \frac{1}{2} \left(1 - \langle \cos^2 \beta \rangle - \langle \cos^2 \theta \rangle + 3 \langle \cos^2 \beta \cos^2 \theta \rangle \right) \quad , \quad (4)$$

and we use this identity to relate R , Q_X and Q_J .

In the zeroth order approximation the alignment in grain axes is independent of the alignment in respect to magnetic field⁴. Therefore

$$\langle \cos^2 \beta \cos^2 \theta \rangle \approx \langle \cos^2 \beta \rangle \langle \cos^2 \theta \rangle \quad , \quad (5)$$

⁴Further in the text we discuss the dependence between the distributions of β and θ due to their dependence on the distribution of J .

and it is easy to see that (Jones & Spitzer 1967)

$$R \approx Q_J \times Q_X \quad . \quad (6)$$

Here, as in Paper I, we consider only oblate grains, because there are indications that aligned grains are oblate rather than prolate (Aitken et al. 1985, Lee & Draine 1985, Hildebrand 1988, Hildebrand & Dragovan 1995). We approximate the grain mantle surface and the core-mantle interface by confocal spheroids and use a_m and b_m ($b_m > a_m$) to denote the mantle semi-axes parallel and perpendicular to the grain symmetry axis, respectively. The corresponding core semi-axes are denoted by a_c and b_c . Then the eccentricity of the core/mantle ($i = c, m$) is

$$e_i = \sqrt{1 - \frac{a_i^2}{b_i^2}} \quad . \quad (7)$$

It may be different for different components of the grain.

3. Gaseous bombardment

It was shown in Jones & Spitzer (1967) that alignment of \mathbf{J} due to paramagnetic relaxation can be described by the Fokker-Planck equation (see Reichl 1980)

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial J_i} \left(\left\langle \frac{\Delta J_i}{\Delta t} \right\rangle f \right) = \frac{1}{2} \frac{\partial^2}{\partial J_i \partial J_j} \left(\left\langle \frac{\Delta J_i \Delta J_j}{\Delta t} \right\rangle f \right) \quad , \quad (8)$$

where f is the distribution function of angular momentum $J \equiv |\mathbf{J}|$, while $\left\langle \frac{\Delta J_i}{\Delta t} \right\rangle$ and $\left\langle \frac{\Delta J_i \Delta J_j}{\Delta t} \right\rangle$ are diffusion coefficients.

The quadratic diffusion coefficients in the grain frame of reference $x^b y^b z^b$ were calculated in Roberge et al. (1993):

$$\left\langle \frac{(\Delta J_z^b)^2}{\Delta t} \right\rangle = \frac{2\sqrt{\pi}}{3} n m b_m^4 v_{th}^3 \Gamma_{\parallel}(e_m) \left(1 + \frac{T_s}{T_g} \right) \quad , \quad (9)$$

$$\left\langle \frac{(\Delta J_i^b)^2}{\Delta t} \right\rangle = \frac{2\sqrt{\pi}}{3} n m b_m^4 v_{th}^3 \Gamma_{\perp}(e_m) \left(1 + \frac{T_s}{T_g} \right) \quad , \quad (10)$$

where $i = x, y$, T_s and T_g are dust and gas temperatures, respectively, and $v_{th} = \sqrt{2kT_g/m}$ is the thermal velocity of gaseous atoms with mass m and concentration n . The coefficients $\Gamma_{\perp}(e_m)$ and $\Gamma_{\parallel}(e_m)$ are geometrical factors

$$\Gamma_{\perp}(e_m) = \frac{3}{32} \{ 7 - e_m^2 + (1 - e_m^2) g_m(e_m) + (1 - 2e_m^2) [1 + e_m^{-2} (1 - [1 - e_m]^2 g(e_m))] \} \quad , \quad (11)$$

$$\Gamma_{\parallel}(e_m) = \frac{3}{16} \{3 + 4(1 - e_m^2)g_m(e_m) - e_m^2(1 - [1 - e_m^2]^2 g(e_m))\}, \quad (12)$$

with

$$g(e_m) = \frac{1}{2e_m} \ln \left(\frac{1 + e_m}{1 - e_m} \right). \quad (13)$$

We cannot use the expressions for $\left\langle \frac{\Delta J_i}{\Delta t} \right\rangle$ obtained in Roberge et al. (1993) as those are found assuming perfect Barnett alignment. Instead we will derive these coefficients from $\left\langle \left(\frac{\Delta J_{ii}}{\Delta t} \right)^2 \right\rangle$ using the approach from Lifshitz & Pitaevskii (1981, Chapter II). In thermal equilibrium, the distribution of angular momentum in grain frame of reference is given by

$$f = \text{const} \times \exp \left(-\frac{J_x^2 + J_y^2}{2I_{\perp} k T_g} - \frac{J_z^2}{2I_z k T_g} \right) \quad (14)$$

and is independent of grain magnetic properties. Then, the terms in the Fokker-Plank equation (8) corresponding to the Davis-Greenstein relaxation, Barnett relaxation and to the gaseous bombardment can be studied separately. Indeed, these are independent processes and in thermodynamic equilibrium fluctuations of each of these parameters should be compensated by the corresponding dissipation. In physical terms this means that neither variations of ambient gaseous pressure, nor variations of grain magnetic properties change the equilibrium distribution (14). Therefore by plugging Eq. (14) in Eq. (8), we obtain

$$-\left\langle \frac{\Delta J_i^b}{\Delta t} \right\rangle = \frac{1}{2I_i k T} \left\langle \frac{(\Delta J_{ii}^b)^2}{\Delta t} \right\rangle J_i \quad (15)$$

Substituting $T_g = T_s$ in Eqs (9) and (10), we get⁵

$$\begin{aligned} \left\langle \frac{\Delta J_z^b}{\Delta t} \right\rangle &= -\frac{4\sqrt{\pi}}{3I_z} nmb_m^4 v_{th} \Gamma_{\parallel}(e_m) J_z^b, \\ \left\langle \frac{\Delta J_j^b}{\Delta t} \right\rangle &= -\frac{4\sqrt{\pi}}{3I_{\perp}} nmb_m^4 v_{th} \Gamma_{\perp}(e_m) J_j^b, \end{aligned} \quad (16)$$

where $j = x, y$. In the limit of perfect Barnett relaxation $J_x^b = J_y^b = 0$, and Eqs (16) reduce to the expressions for the diffusion coefficients obtained in Roberge et al. (1993).

Introducing time-scales

$$\begin{aligned} t_{gas1} &= \frac{3I_z}{4\sqrt{\pi}} \frac{1}{nmb_m^4 v_{th} \Gamma_{\parallel}(e_m)}, \\ t_{gas2} &= \frac{3I_z}{4\sqrt{\pi}} \frac{1}{nmb_m^4 v_{th} \Gamma_{\perp}(e_m)}, \end{aligned} \quad (17)$$

⁵Note that $\left\langle \frac{\Delta J_i^b}{\Delta t} \right\rangle$ is independent of grain temperature. Therefore Eqs (16) present a general form of $\left\langle \frac{\Delta J_i^b}{\Delta t} \right\rangle$ valid for any T_g and T_d . Another way of deriving Eqs (16) is presented in LR.

it is possible to rewrite expressions (16) as follows

$$\begin{aligned}\left\langle \frac{\Delta J_z^b}{\Delta t} \right\rangle &= -J_z^b/t_{gas1} \quad , \\ \left\langle \frac{\Delta J_j^b}{\Delta t} \right\rangle &= -J_j^b/t_{gas2} \quad ,\end{aligned}\tag{18}$$

where $j = x, y$.

The component of the moment of major inertia I_z can be as much as two times greater than I_\perp for extremely flat grains. At the same time Γ_\parallel is only slightly greater than Γ_\perp . Therefore for any eccentricity, $t_{gas1} > t_{gas2}$ (see Fig. 2). This corresponds to an intuitive perception that an oblate grain rotating about its axis of major inertia experiences less friction than the one rotating about a perpendicular axis. Indeed, the friction torque is proportional to grain’s angular velocity $\sim J/I_i$, where $i = x, y$ or z depending on the axis of rotation. Thus for a fixed J the drag is inversely proportional to the moment of inertia in accordance with Eq. (17).

In general, due to the difference between t_{gas1} and t_{gas2} the vector of the dissipation angular momentum is not directed along \mathbf{J} . Instead it lies in the plane defined by \mathbf{J} and axis of major inertia. If grain rotational temperature is determined by gas-grain collisions, the difference in “parallel” and “perpendicular” damping is compensated by the difference in excitation of rotation through the quadratic terms. However, for suprathermal rotation, this difference ($t_{gas1} > t_{gas2}$!) should provide alignment of \mathbf{J} with the axis of major inertia. For paramagnetic grains, this mechanism is less efficient as compared with the Barnett relaxation (see Section 6.2), but there may be situations, where it is important. A further discussion of this interesting phenomenon will be given elsewhere.

4. Averaging over precession

Grains perform complex motion: for one thing, the grain axis of major inertia \mathbf{z}^b precesses about \mathbf{J} , for another thing, \mathbf{J} precesses about magnetic field. The period of the latter is much greater ($\sim 10^{10}$ times) than the period of the former precession, but is small compared to the gaseous damping time. Therefore, following usual approach adopted in theoretical mechanics to “fast” tops (see Landau & Lifshitz 1976), we shall at first neglect

ambient magnetic field⁶ and consider averaging over free precession of \mathbf{z}^b about the direction of the angular momentum \mathbf{J} . Then the averaging over Larmor precession (i.e. precession of \mathbf{J} about magnetic field) will be performed.

4.1. Averaging over precession about the axis of major inertia

The angular momentum rapidly precesses in grain body axes and the angular velocity of such precession is of the order of grain angular velocity.

It is convenient to introduce a new Cartesian reference system x, y, z , where \mathbf{z} is directed along \mathbf{J} and \mathbf{x} and \mathbf{y} lie in the plane normal to \mathbf{J} . The relative orientation of xyz and the body frame is given by Eulerian angles (see Fig. 1). For an oblate spheroid, x^b -axis can be taken along the lines of nodes, i.e. $\psi = 0$, which greatly simplifies further calculations.

The increments of the angular momentum along x^b , y^b , and z^b axes are related to the increments along x , y , and z axes in the following way:

$$\begin{aligned}\Delta J_x &= \Delta J_x^b \cos \phi - \Delta J_y^b \cos \theta \sin \phi + \Delta J_z^b \sin \theta \sin \phi \quad , \\ \Delta J_y &= \Delta J_x^b \sin \phi + \Delta J_y^b \cos \theta \cos \phi - \Delta J_z^b \sin \theta \sin \phi \quad , \\ \Delta J_z &= \Delta J_y^b \sin \beta + \Delta J_z^b \cos \beta \quad .\end{aligned}\tag{19}$$

This approach can provide the “precession averaged” coefficients for arbitrary grains. However, to simplify the resulting formulae we account for the grain rotational symmetry and introduce the following notation for the coefficients in the body axes

$$\begin{aligned}\left\langle \frac{(\Delta J_\perp)^2}{\Delta t} \right\rangle &= \left\langle \frac{(\Delta J_x^b)^2}{\Delta t} \right\rangle = \left\langle \frac{(\Delta J_y^b)^2}{\Delta t} \right\rangle \quad , \\ \left\langle \frac{(\Delta J_\parallel)^2}{\Delta t} \right\rangle &= \left\langle \frac{(\Delta J_z^b)^2}{\Delta t} \right\rangle \quad .\end{aligned}\tag{20}$$

Averaging over ϕ gives

$$\begin{aligned}\left\langle \frac{\Delta J_i}{\Delta t} \right\rangle &= 0 \quad , \\ \left\langle \frac{\Delta J_z}{\Delta t} \right\rangle &= \left\langle \frac{\Delta J_z^b}{\Delta t} \right\rangle \cos \theta + \left\langle \frac{\Delta J_y^b}{\Delta t} \right\rangle \sin \theta \quad ,\end{aligned}\tag{21}$$

⁶We neglect the magnetic field while averaging, but account for its action in terms of decreasing the value of angular momentum.

where $i = x, y$. Using Eq. (18), it is possible to rewrite Eq. (21) as

$$\begin{aligned}\left\langle \frac{\Delta J_i}{\Delta t} \right\rangle &= 0 \quad , \\ \left\langle \frac{\Delta J_z}{\Delta t} \right\rangle &= -\frac{J_z^b}{t_{gas1}} \cos \theta - \frac{J_y^b}{t_{gas2}} \sin \theta \quad ,\end{aligned}\tag{22}$$

which shows, that the averaged vector of the gaseous damping is directed along \mathbf{J}^7 . As we have chosen x^b -axis along the lines of nodes, the angular momentum is

$$\mathbf{J} = J_y^b \mathbf{i}_y^b + J_z^b \mathbf{i}_z^b \quad ,\tag{23}$$

where \mathbf{i}_y^b and \mathbf{i}_z^b are unit vectors along y^b and z^b -axes, respectively, in the grain frame of reference. It is easy to see, that

$$\tan \theta = \frac{J_y^b}{J_z^b} \quad .\tag{24}$$

Combining Eqs (22), (23) and (24), it is possible to introduce the effective damping time

$$t_{\text{eff}} = t_{gas1} \cdot \xi(\theta) \quad ,\tag{25}$$

where

$$\xi(\theta) = \frac{1 + \tan^2 \theta}{1 + \frac{t_{gas1}}{t_{gas2}} \tan^2 \theta} \quad .\tag{26}$$

According to Fig. 2, $\frac{t_{gas1}}{t_{gas2}} > 1$, and therefore $t_{\text{eff}} < t_{gas1}$. However for grains of small eccentricities and/or $\theta \ll 1$, the difference between t_{eff} and t_{gas1} is small. Using t_{eff} it is possible to rewrite Eq. (22) in the following way

$$\left\langle \frac{\Delta J_z}{\Delta t} \right\rangle = -\frac{J_z}{t_{\text{eff}}} \quad .\tag{27}$$

After simple algebra we obtain the “precession averaged” diagonal quadratic coefficients of the Fokker-Planck equation:

$$\begin{aligned}\left\langle \frac{(\Delta J_i)^2}{\Delta t} \right\rangle &= \frac{1}{2} \left(\left\langle \frac{(\Delta J_{\perp})^2}{\Delta t} \right\rangle [1 + \cos^2 \theta] + \left\langle \frac{(\Delta J_{\parallel})^2}{\Delta t} \right\rangle \sin^2 \theta \right) \quad , \\ \left\langle \frac{(\Delta J_z)^2}{\Delta t} \right\rangle &= \left\langle \frac{(\Delta J_{\parallel})^2}{\Delta t} \right\rangle \cos^2 \theta + \left\langle \frac{(\Delta J_{\perp})^2}{\Delta t} \right\rangle \sin^2 \theta \quad ,\end{aligned}\tag{28}$$

⁷The component of damping force orthogonal to \mathbf{J} produces insignificant nutations of \mathbf{J} . This is in contrast to its action in the grain frame of reference, where it can contribute to the alignment of \mathbf{J} with the axis of major inertia.

where $i = x, y$ and all $\left\langle \frac{\Delta J_i \Delta J_j}{\Delta t} \right\rangle$, $i \neq j$, vanish.

The “precession averaging” we perform here is similar to the “Larmor averaging” used in Roberge et al. (1993). The difference is that we allow for the increments of angular momentum along all axes, while only increments along z^b -axis were accounted for in Roberge et al. (1993).

4.2. Averaging over precession about magnetic field

Due to the Barnett effect, a spinning grain develops magnetic moment anti-parallel to grain angular velocity (Dolginov & Mytrophanov 1976, Purcell 1979). This magnetic moment interacts with the ambient magnetic field and causes grain precession on the time-scale of several years.

Although the Larmor precession is much slower than the precession of \mathbf{z}^b about \mathbf{J} , it is fast compared to the rate of gaseous damping and paramagnetic alignment. Therefore the “Larmor averaging” is necessary. This averaging is very similar to the one used above.

The Larmor precession is characterized by angle β . Let us introduce a system of reference $x_0 y_0 z_0$ with z_0 -axis along magnetic field and x_0 and y_0 axes lying in the plane perpendicular to the field. It is easy to obtain quadratic coefficients

$$\begin{aligned} \left\langle \frac{(\Delta J_i)^2}{\Delta t} \right\rangle &= \frac{1}{2} \left\langle \frac{(\Delta J_\perp)^2}{\Delta t} \right\rangle \left(\frac{1}{2} [1 + \cos^2 \theta] [1 + \cos^2 \beta] + \sin^2 \beta \sin^2 \theta \right) \\ &\quad + \frac{1}{2} \left\langle \frac{(\Delta J_\parallel)^2}{\Delta t} \right\rangle \left(\frac{1}{2} \sin^2 \theta [1 + \cos^2 \beta] + \cos^2 \theta \sin^2 \beta \right) , \\ \left\langle \frac{(\Delta J_{z_0})^2}{\Delta t} \right\rangle &= \left\langle \frac{(\Delta J_\parallel)^2}{\Delta t} \right\rangle \left(\cos^2 \theta \cos^2 \beta + \frac{1}{2} \sin^2 \theta \sin^2 \beta \right) \\ &\quad + \left\langle \frac{(\Delta J_\perp)^2}{\Delta t} \right\rangle \left(\sin^2 \theta \cos^2 \beta + \frac{1}{2} [1 + \cos^2 \theta] \sin^2 \beta \right) , \end{aligned} \quad (29)$$

where $i = x_0, y_0$. The substitution of expressions for $\left\langle \frac{(\Delta J_\perp)^2}{\Delta t} \right\rangle$, $\left\langle \frac{(\Delta J_\parallel)^2}{\Delta t} \right\rangle$ in Eq. (29) provides

$$\begin{aligned} A &= \frac{\sqrt{\pi}}{3} n m b_m^4 v_{th}^3 \left(1 + \frac{T_s}{T_g} \right) , \\ \left\langle \frac{(\Delta J_i)^2}{\Delta t} \right\rangle &= A \Gamma_\perp (e_m) \left[\frac{1}{2} [1 + \cos^2 \theta] [1 + \cos^2 \beta] + \sin^2 \beta \sin^2 \theta \right] , \end{aligned}$$

$$\begin{aligned}
& + A\Gamma_{\parallel}(e_m) \left[\frac{1}{2} \sin^2 \theta [1 + \cos^2 \beta] + \cos^2 \theta \sin^2 \beta \right] \\
\left\langle \frac{(\Delta J_{z_0})^2}{\Delta t} \right\rangle & = 2A\Gamma_{\parallel}(e_m) \left[\cos^2 \theta \cos^2 \beta + \frac{1}{2} \sin^2 \theta \sin^2 \beta \right] \\
& + 2A\Gamma_{\perp}(e_m) \left[\sin^2 \theta \cos^2 \beta + \frac{1}{2} [1 + \cos^2 \theta] \sin^2 \beta \right] \quad , \quad (30)
\end{aligned}$$

where $i = x_0, y_0$. If $\theta = 0$, it is easy to see that our coefficients coincide with those in Roberge et al. (1993).

According to Eq. (27), the damping is given by a vector anti-parallel to \mathbf{J} . Therefore

$$\left\langle \frac{\Delta J_i}{\Delta t} \right\rangle = -\frac{J_i}{t_{\text{eff}}} \quad , \quad (31)$$

where $i = x_0, y_0, z_0$.

The coefficients above can be used to solve the Fokker-Planck equation numerically similarly to what was done in Roberge et al. (1993). Instead we will use a perturbative approach similar to one introduced in Paper I. This approach was shown to provide good accuracy when the Barnett relaxation is complete and therefore we use it here. In future we plan to compare our analytical results with direct numerical simulations when these simulations become available.

5. Paramagnetic relaxation

The diffusion coefficients above characterize gas-grain interactions. The effect of magnetic field with intensity B on grains is imprinted through the “magnetic” coefficients, their simplest form in $x_0 y_0 z_0$ reference frame is (see Jones & Spitzer 1967)

$$\left\langle \frac{\Delta J_j}{\Delta t} \right\rangle_{\text{mag}} = -\frac{J_j}{t_{\text{mag}}} \quad , \quad (32)$$

$$\left\langle \frac{\Delta J_{z_0}}{\Delta t} \right\rangle_{\text{mag}} = 0 \quad , \quad (33)$$

where $j = x_0, y_0$,

$$t_{\text{mag}} = \frac{I_z^b}{\kappa V B^2} \quad , \quad (34)$$

with V denoting grain volume and $\kappa \approx 2.5 \cdot 10^{-12} T_s^{-1}$ s used for slow rotation (Spitzer 1978). Similarly

$$\left\langle \frac{(\Delta J_j)^2}{\Delta t} \right\rangle_{\text{mag}} = 2kT_s V B^2 \kappa \quad , \quad (35)$$

$$\left\langle \frac{(\Delta J_{z_0})^2}{\Delta t} \right\rangle_{\text{mag}} = 0 \quad , \quad (36)$$

where $j = x_0, y_0$. Note, that Eqs (33) and (36) reflect the fact that magnetic field does not slow down grains rotating about the field.

Only the component of magnetic field perpendicular to \mathbf{J} contributes to the Davis-Greenstein relaxation. The component parallel to \mathbf{J} contributes to the relaxation in the grain frame of reference.⁸

6. Perturbative approach

6.1. Small parameters

It is well known that asymptotic results are usually attainable if there is a small parameter in a model. The difficulty in direct solving the Fokker-Planck equation comes from the fact that both $\left\langle \frac{\Delta J_i}{\Delta t} \right\rangle$ and $\left\langle \frac{(\Delta J_i)^2}{\Delta t} \right\rangle$ depend on two variables, namely, θ and β .

In general, Γ_{\parallel} differs from Γ_{\perp} , but according to Paper I, their ratio is close to unity and therefore

$$\gamma = 1 - \frac{\Gamma_{\perp}}{\Gamma_{\parallel}} \quad (37)$$

does not exceed 0.2. Using γ , it is possible to rewrite Eqs (30) so that terms containing γ are clearly separated

$$\begin{aligned} \left\langle \frac{(\Delta J_j)^2}{\Delta t} \right\rangle &= 2A\Gamma_{\parallel}[1 - \gamma\eta_2(\beta, \theta)] \quad , \\ \left\langle \frac{(\Delta J_{z_0})^2}{\Delta t} \right\rangle &= 2A\Gamma_{\parallel}[1 - \gamma\eta_1(\beta, \theta)] \quad , \end{aligned} \quad (38)$$

⁸This process can be called Jones & Spitzer relaxation, as it was first mentioned in Jones & Spitzer (1967).

where $j = x_0, y_0$ and

$$\eta_1(\beta, \theta) = \sin^2 \theta + \sin^2 \theta (\cos^2 \beta - 0.5 \sin^2 \beta) \quad , \quad (39)$$

$$\eta_2(\beta, \theta) = \frac{1}{2}([1 + \cos^2 \theta] + \sin^2 \theta [\cos^2 \beta - 0.5 \sin^2 \beta]) \quad . \quad (40)$$

It is easy to see that both η_1 and η_2 do not exceed unity for all possible values θ and β . Therefore the problem of perturbative treatment of these coefficients is very similar to that used in Paper I.

A new feature of our present study as compared with Paper I is the dependence of the linear coefficient given by Eq. (27) on θ . When $\theta \ll 1$, the variations of $\xi(\theta)$ (see Eq. (25)) are small. The upper bound for θ can be found by comparing our predictions with direct numerical simulations whenever such simulations become available. Therefore for the time being, we study paramagnetic alignment assuming that θ is our second small parameter. Fortunately, for many cases of practical importance θ is small (LR).

6.2. Asymptotic of the Barnett relaxation

If $\theta \neq 0$, \mathbf{J} precesses in body coordinates. Purcell (1979) was the first to realize that this must entail internal dissipation of energy on relatively short time scales. According to Purcell (1979) the most important mechanism of internal dissipation is the Barnett dissipation⁹. This relaxation arises from alternating magnetization caused by precessing Barnett moment (see previous section). For a thermally rotating grain of size $a = 10^{-5}$ cm, the Barnett-equivalent magnetic field is $\omega/\mu_r \approx 10^{-2} a_{-5}^{5/2}$ G, which is substantially stronger than interstellar magnetic field.¹⁰ Therefore the rate of relaxation, which scales as B^2 , is nearly 10^6 times faster than the paramagnetic alignment.

⁹Our preliminary study shows that this is not always true and Purcell (1979) underestimated the efficiency of inelastic dissipation. This issue, however, does not change our conclusions here and we will discuss inelastic dissipation elsewhere.

¹⁰ In some circumstances, e.g. in stellar atmospheres the external magnetic field may become greater than the “Barnett induced” one. In this case the relaxation is mainly caused by the component of external magnetic field parallel to \mathbf{J} . This, however, may not alter our results here, provided that the relaxation happens on the time-scale much less than the time of gaseous damping.

An elaborate study of the partial Barnett alignment was done in LR for different ratios of Barnett and gaseous damping times. Here we consider the alignment when this ratio approaches zero. In this regime the value of θ is determined only by thermal fluctuations within grain material and for a fixed angular momentum, is given by the Boltzmann distribution (LR)

$$f(\theta) = \text{const} \times \sin \theta \exp [-E_{\text{rot}}(\theta)/kT_s] \quad , \quad (41)$$

where const is determined by normalization, and grain rotational energy is

$$E_{\text{rot}}(\theta) = \frac{J^2}{2I_z} \left[1 + (h - 1) \sin^2 \theta \right] \quad , \quad (42)$$

where $h = I_z/I_{\perp}$.

The mean value of $\cos \theta$ can be found as

$$\langle \cos \theta \rangle_J = \frac{\int_0^\pi \cos^2 \theta \sin \theta f(\theta) d\theta}{\int_0^\pi \sin \theta f(\theta) d\theta} \quad , \quad (43)$$

where the subscript J indicates that the value for is calculated for an individual grain with a fixed angular momentum.

The value of \mathbf{J} varies for different grains within an ensemble. To find the mean value of $\cos \theta$ for the ensemble of grains, $\langle \cos \theta \rangle$, one has to average over the distribution of J . However, numerical studies of Barnett alignment in LR95 have shown that with a high degree of accuracy one can find $\langle \cos \theta \rangle$ by substituting the mean value of angular momentum corresponding to rotational temperature T in Eq. (43). This phenomenological fact greatly simplifies our treatment and here we want to provide a theoretical justification for it by studying asymptotics of the Barnett relaxation in the limit $\theta \ll 1$.

Consider the evolution of \mathbf{J} in xyz reference system. The contribution of the Barnett related coefficients in this system is zero, as the Barnett relaxation does not change the value of J . Therefore it is possible to formulate the Fokker-Planck equation with the linear coefficient given by Eq. (27) and the quadratic coefficient given by the sum of the quadratic coefficients (28):

$$\left\langle \frac{(\Delta J)^2}{\Delta t} \right\rangle = 3A\Gamma_{\parallel} [1 - \gamma/3] \quad . \quad (44)$$

For small θ , the θ -dependence of the linear coefficient is weak (27). Thus we fix angle $\theta = \theta_i \ll 1$ and find

$$F(J) = \text{const} \times \exp \left(-\frac{J^2}{3kT_m I_z \xi(\theta_i)(1 - \gamma/3)} \right) \quad , \quad (45)$$

where

$$T_m = 0.5(T_m + T_g) \quad . \quad (46)$$

A study of Eq. (45) reveals only a weak dependence of J on θ and this corresponds to calculations in LR performed for different distributions of θ . This means that our perturbative solution describes essential physics for $\theta \ll 1$.

Further on in the paper we use the phenomenological result of LR and calculate the measure of internal alignment Q_J by substituting the ensemble averaged value of J in Eq. (43). In the absence of paramagnetic relaxation, the Maxwellian mean value of angular momentum should be used (Landau & Lifshitz 1980):

$$J^2 = kT_m(2I_\perp + I_z) \quad . \quad (47)$$

In the presence of paramagnetic dissipation, this value can still be used as the zeroth approximation for θ_0 to obtain the zeroth order approximation for t_{eff} and diffusion coefficients.

In general, paramagnetic relaxation diminishes the value of J . Paramagnetic alignment decreases the component of \mathbf{J} perpendicular to magnetic field and does not affect the component of \mathbf{J} parallel to the field. Therefore it is reasonable to conjecture that the component of \mathbf{J} parallel to magnetic field stays Maxwellian and

$$J^2 = \frac{kT_m(2I_\perp + I_z)}{3 \langle \cos^2 \beta \rangle} \quad (48)$$

can be substituted instead of Eq. (43) as a mean value of J^2 . Equations (48), (41), (42) and (43) enable one to obtain higher order iterations of the alignment measures.

6.3. Iterations

We start with assuming $\gamma = 0$ and $\theta = \theta_0$. Then variables in Eq. (8) can be separated. The stationary equation for the z component is

$$\frac{1}{2} \frac{\partial^2}{\partial J_{z_0}^2} \left(\left\langle \frac{(\Delta J_{z_0})^2}{\Delta t} \right\rangle f_{z_0} \right) - \frac{\partial}{\partial J_{z_0}} \left(\left\langle \frac{\Delta J_{z_0}}{\Delta t} \right\rangle f_{z_0} \right) = 0 \quad , \quad (49)$$

which has the solution

$$\ln f_{z_0} = - \frac{J_{z_0}^2}{t_{\text{eff}} \left\langle \frac{(\Delta J_{z_0})^2}{\Delta t} \right\rangle} + \text{const}_1 \quad . \quad (50)$$

In the case of x_0 and y_0 components, one has to account for paramagnetic relaxation and the solutions take the form

$$\ln f_j = -\frac{J_j^2}{\langle(\Delta J_j)^2\rangle + \langle(\Delta J_j)^2\rangle_{\text{mag}}} \left(\frac{1}{t_{\text{eff}}} + \frac{1}{t_{\text{mag}}} \right) + \text{const}_2 \quad . \quad (51)$$

To characterize the relative importance of magnetic torque, we define

$$\delta_i = \frac{\left\langle \frac{\Delta J_j}{\Delta t} \right\rangle_{\text{mag}}}{\left\langle \frac{\Delta J_j}{\Delta t} \right\rangle} = \frac{t_{\text{eff}}}{t_{\text{mag}}} = \frac{3}{4\sqrt{\pi}} \frac{\kappa V B^2}{nm v_{th} b_m^4 \Gamma_{\parallel}} \xi(\theta_i) \quad . \quad (52)$$

As $\xi(\theta) < 1$ (see Fig. 3), Eq. (52) shows that the incomplete alignment increases randomization for oblate grains.

For $\gamma = 0$, $\theta = \theta_0$ and $\delta_i = \delta_0$, the problem is similar to that discussed in Jones & Spitzer (1967). The solution is also similar

$$\cos^2 \theta_0 = \frac{1}{1 - \aleph_0^2} \left[1 - \frac{\aleph_0}{\sqrt{1 - \aleph_0^2}} \arcsin \sqrt{1 - \aleph_0^2} \right] \quad , \quad (53)$$

where

$$\aleph_0^2 = \frac{1 + \delta_0 T_s / T_m}{1 + \delta_0} \quad . \quad (54)$$

For higher iterations modified diffusion coefficients are

$$t_{\text{eff}} \left\langle \frac{(\Delta J_{z_0})^2}{\Delta t} \right\rangle = 2kT_m I_z \xi(\theta_i) (1 - \gamma \eta_1(\beta_i, \theta_i)) \quad , \quad (55)$$

$$t_{\text{eff}} \left(\left\langle \frac{(\Delta J_j)^2}{\Delta t} \right\rangle + \left\langle \frac{(\Delta J_j^2)}{\Delta t} \right\rangle_{\text{mag}} \right) = 2kI_{z_0} \xi(\theta_i) (T_m + \delta_i T_d) \left(1 - \gamma \frac{1 + \eta_2(\beta_i, \theta_i)}{1 + \delta_i T_d / T_m} \right) \quad , \quad (56)$$

$$t_{\text{eff}} \left\langle \frac{(\Delta J_j)^2}{\Delta t} \right\rangle_{\text{mag}} = 2kT_s \delta_1 I_z^b \quad , \quad (57)$$

where $j = x_0, y_0$.

Thus, the distribution function for the i -th iteration is

$$f = f_{x_0} f_{y_0} f_{z_0} = \text{const} \cdot \exp \left\{ \frac{J^2 (1 - \cos^2 \beta [1 - \aleph_i^2])}{2k\xi(\theta_i) I_z T_{\text{av}} (1 - \gamma W_i)} \right\} \quad , \quad (58)$$

where

$$T_{\text{av}} = \frac{T_m + T_s \delta_{i-1}}{1 + \delta_{i-1}} \quad , \quad (59)$$

$$W_i = \frac{1 + \eta_2(\beta_{i-1}, \theta_{i-1})}{1 + \delta_{i-1}T_d/T_m}. \quad (60)$$

The solution for $\langle \cos^2 \beta_i \rangle$ can be obtained by substituting

$$\aleph_i = \frac{T_{\text{av}}(1 - \gamma W_i)}{T_m[1 - \gamma\eta_1(\beta_{i-1}, \theta_{i-1})]} \quad (61)$$

in Eq. (53) for \aleph_0 .

Using Eqs (43) and (53), we can find i -th iterations of Q_X and Q_J (see Eqs (2) and (3)). Then Eq. (6) gives us the Rayleigh reduction factor and this solves the problem. It is possible to show that the corresponding formal series converge and the error of the zero approximation does not exceed ten percent for a wide range of ratios of magnetic to damping times and grain to gas temperatures. We stress “formal”, however, as in our perturbative treatment we substitute particular values of θ_i and β_i instead of the distributions of those angles. Therefore a direct check of our results by testing against numerical simulations for a wide range of δ_i and T_d/T_g is necessary. Although, the corresponding numerical study is a challenging problem far more involved than the numerical simulations performed so far we hope that it can be accomplished (Roberge & Lazarian 1997, in preparation).

6.4. Superparamagnetic grains

An important case of grains that can be paramagnetically aligned is “supergrains” (grains with superparamagnetic or superferromagnetic properties) suggested in Jones & Spitzer (1967) (see also Draine 1996). Indeed, ordinary paramagnetic grains are only marginally aligned in typical interstellar conditions¹¹ and therefore cannot account for observed polarization. On the contrary, “supergrains” may be efficiently aligned and can account for the peculiarities of the polarization curve (Mathis 1986, Martin see also Goodman & Whittet 1996). Such grains rapidly dissipate their energy by interacting with external magnetic field, and their ratio of gaseous to magnetic damping time is much greater than unity. Then δ_i given by Eq. (52) is $\gg 1$ for any value of θ , and Eq. (59)

¹¹Here we speak about alignment of grains with $a > 10^{-5}$ cm, which are responsible for the observed polarization (Kim & Martin 1995). One may erroneously assume that due to the dependence of δ_i on grain size (see Eq. (52)) small grain should be efficiently aligned. For those grains, however, one has to account for disorientation of grains through thermal emission and this decreases δ_i .

provides $T_{av} \approx T_d$, i.e. the dependence on θ cancels out. As a result, for $\gamma = 0$, Eq. (61) gives $\aleph \approx T_d/T_m$, which coincides with the result obtained in Jones & Spitzer (1967) for spherical superparamagnetic grains.

For $\gamma \neq 0$, the iterations involve only one small parameter and the problem is similar to that treated in Paper I. The parameter \aleph_i can be written in the following way

$$\aleph_i^2 \approx \frac{T_{av}}{T_m[1 - \gamma\xi(\beta_i, \theta_i)]} \approx \frac{T_{av}}{T_m}[1 + \gamma\xi(\beta_i, \theta_i)] \quad , \quad (62)$$

where we disregarded γW_i term on the account that $W_i \ll 1$ when $\delta_i \gg 1$. For any θ , $\xi(\theta) < 1$ and formally there is no difference between Eq. (62) and eq. 40 in Paper I. The latter expression for \aleph resulted in the iteration procedure that was proved accurate by numerical simulations¹² (Roberge (1996), DeGraff, Roberge & Flaherty (1997)). Therefore we believe that at least for superparamagnetic grains our analytical treatment is accurate.

7. Discussion

In comparing our present results with those obtained in Paper I we want to define clearly the ranges of applicability of the corresponding formalisms.

Our results in Paper I are applicable to grains, which have rotational temperature much greater than the temperature of grain material. In this limit the alignment of \mathbf{J} in the grain frame of reference is almost perfect and the measure of alignment can be obtained through the iteration procedure.

Apart from a trivial case of $T_g \gg T_d$ it is likely that the treatment presented in Paper I is applicable if active sites of H_2 formation cover the entire surface of grains and therefore H_2 formation is essentially stochastic (Cugnon 1985). Such a chaotic formation of H_2 molecules can make grains “rotationally hot” ($T_m \gg T_d$). Cosmic rays can also spin up grains in particular regions through the process of momentum deposition described in Purcell & Spitzer (1971), though really huge fluxes of cosmic rays are needed for the purpose.

As compared with Paper I, the formalism presented in the present paper is more versatile. Indeed, it allows to find the measure of the Davis-Greenstein alignment for finite

¹² According to private communication by W. Roberge some deviations were observed in the limit of very small δ_i . Such δ_i are of minor practical importance, however.

T_m/T_g ratios. Unlike all earlier papers on the Davis-Greenstein alignment the present one accounts for a recently discovered phenomenon of incomplete internal relaxation. This enables one to find the Rayleigh reduction factor for oblate grains with high accuracy.

Our results are especially important for superparamagnetic and superferromagnetic grains. Fortunately, for such grains the formal treatment of the problem of \mathbf{J} alignment in respect to magnetic field is similar to that in Paper I. The accuracy of the latter treatment was proved through numerical simulations and this makes us optimistic about the applicability of our results to “supergrains” for a wide range of grain and gas temperatures.

For ordinary paramagnetic grains, our analytical results are obtained in the limit of small deviations of \mathbf{J} from the axis of major inertia. Although formally it is possible to iterate for arbitrary deviations and obtain converging series for the solution, we believe that more numerical studies are needed to test our results for finite θ .

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Figure Captions

Fig.1 Euler angles are shown on this figure. The z^b axis is directed along the axis of major inertia, while z -axis is directed along the vector of angular momentum. Similar transformations involving Euler angles are used to relate diffusion coefficients in the course of Larmor averaging. In the latter case the equivalent of z^b -axis is directed along \mathbf{J} , while the equivalent of z -axis along magnetic field. Naturally, in the angle between these two axes is not θ , but β . Postscript file of this figure was given to us by Wayne Roberge.

Fig.2 The ratio of t_{gas1}/t_{gas2} as a function of grain eccentricity. Zero eccentricity corresponds to spherical grains and eccentricity $\rightarrow 1$ to flakes.

Fig.3 The ratio of t_{eff}/t_{gas1} as a function of θ . For sufficiently small θ this ratio is close to unity.





